Legendrian and transverse knots in the light of Heegaard Floer Homology

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PhD Thesis submitted to

Eötvös Loránd University
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Budapest
2009

1. Introduction

Legendrian and transverse knot theory has been shaped by advances in convex surface theory [7] (showing that different looking objects are actually equivalent) and by the introduction of various invariants of these knots — proving that different looking objects are, in fact, different. Examples of such invariants are provided by Chekanov's differential graded algebras and contact homology [2, 3]. More recently, Heegaard Floer homology provided various sets of invariants: for knots in the standard contact 3–sphere the combinatorial construction of knot Floer homology through grid diagrams [16, 24], for null–homologous knots in general contact 3–manifolds the Legendrian invariant of [13] and for general Legendrian knots the sutured invariant of the knot complement [10]. In this dissertation we study these invariants to get a better understanding of Legendrian and transverse knots.

1.1. **Heegaard Floer theories.** *Heegaard Floer homologies*, (Ozsváth-Szabó, [19, 20, 22]) the recently-discovered invariants for 3- and 4-manifolds, come from an application of Lagrangian Floer homology to spaces associated to Heegaard diagrams. Although this theory is conjecturally isomorphic to Seiberg-Witten theory, it is more topological and combinatorial in its flavor and thus easier to work with in certain contexts. These homologies admit generalizations and refinements for knots (Ozsváth-Szabó [18] and Rasmussen [26]) and links (Ozsváth-Szabó [23]) in 3–manifolds and for non-closed 3–manifolds with certain boundary conditions (Juhász [11]), called sutured Floer homology. The tools used to define the link-version were later applied to define a completely combinatorial version of knot Floer homology in the 3–sphere.

1.2. **Contact 3–manifolds.** Although contact geometry was born in the late 19th century in the work of Sophus Lie, it has just recently started to develop rapidly, with the discovery of convex surface theory and by recognizing their role in other parts of topology. For example Property P for knots —a possible first step for resolving the Poincaré conjecture— was proved using contact 3–manifolds (Kronheimer-Mrowka [12]). Also, the fact that Heegaard Floer homology determines the Seifert genus of a knot was first proved with the help of contact 3–manifolds (Ozsváth-Szabó [17]). Being the natural boundaries of Stein domains, the use of contact 3–manifolds resulted in a topological description of Stein-manifolds. A *contact structure* on an oriented 3–manifold is a totally non-integrable plane field. In other words it is a plane distribution that is not everywhere tangent to any open embedded surface. Any 3–manifold admits a contact structure (Martinet [14]). It is more subtle though to understand the set of all different contact structures on a given 3–manifold. One way to understand them is by examining lower dimension submanifolds that respect the structure in a way. The 2 dimensional such submanifolds are called *convex surfaces*. These are surfaces with a vectorfield in their neighborhood which is transverse to the surface and whose flow preserves the contact plane distribution. Contact structures in the neighborhood of a convex surface are determined by a set of closed curves (*dividing curves*) on the surface (Giroux [9]). Thus convex surfaces became the right boundary conditions for contact 3–manifolds. In Heegaard Floer homology contact invariants were defined for contact 3–manifolds without (Ozsváth-Szabó [21]) or with (Honda-Kazez-Matic [10]) boundary. These invariants had many applications the most recent is a new proof for the fact that a contact 3–manifold having Giroux torsion cannot be Stein-fillable (Ghiggini-Honda-Van Horn-Morris [8]).

1.3. **Legendrian and transverse knots.** There are two ways for a one dimensional submanifold to respects the contact structure. Its tangents can entirely lie in the plane distribution, in which case the knot is called *Legendrian knot*, or if the tangents are transverse to the planes, the knot is then called a *transverse knot*. A Legendrian knot with a given knot type has two classical invariants: its Thurston-Bennequin number and its rotation number. While for transverse knots there is only one invariant; the self-linking number. The problem of classifying Legendrian (transverse) knots up to Legendrian (transverse) isotopy naturally leads to the question whether these invariants classify Legendrian (transverse) knots. A knot type is called *Legendrian (transverse) simple* if any two realizations of it with equal classical invariants are Legendrian (transverse) isotopic. The unknot (Eliashberg-Fraser [4]), torus knots and the figure-eight knot (Etnyre-Honda [7]) were proved to be both Legendrian and transversely simple. By constructing a new invariant for Legendrian knots, Chekanov [2] showed that not all knots are Legendrian simple, in particular he proved that the knot 5_2 is not Legendrian simple. Later many other Legendrian non-simple knots were found (Epstein-Fuchs-Meyer [5] and Ng [15]). The case for transverse knots turned out to be harder. Birman and Menasco [1], and Etnyre and Honda [6] constructed families of transversely non-simple knots using braid and convex surface theory. The Legendrian invariant in the combinatorial Floer homology provided another tool to construct transversely non-simple knots (Ng-Ozsváth-Thurston [16]).

Using the language of Heegaard Floer homology recently three different invariants were defined for Legendrian and transverse knots. One in the combinatorial settings of knot Floer homology for the 3–sphere [25]: $\widehat{\lambda}$, one in knot Floer homology for a general contact 3–manifold [13]: $\widehat{\mathcal{L}}$ and one defined as the contact invariant associated to the knot-complement: EH.

2. Thesises

In the dissertation I used Heegaard Floer homology to prove theorems about Legendrian and transverse knots.

Similarly to the smooth case there is a well defined notion of connect summing Legendrian or transverse knots.

Theorem 2.1 (Vértesi [29]). *Let L_1 and L_2 be (oriented) Legendrian knots of topological type K_1 and K_2. Then there is an isomorphism*

$$\mathrm{HFK}^-(m(K_1)) \otimes_{\mathbb{Z}_2[U]} \mathrm{HFK}^-(m(K_2)) \to \mathrm{HFK}^-(m(K_1 \# K_2))$$

which maps $\lambda_+(L_1) \otimes \lambda_+(L_2)$ to $\lambda_+(L_1 \# L_2)$. Similar statement holds for the λ_--invariant.

Corollary 2.2 (Vértesi [29]). *Let L_1 and L_2 be (oriented) Legendrian knots of topological type K_1 and K_2. Then there is an isomorphism*

$$\widehat{\mathrm{HFK}}(m(K_1)) \otimes_{\mathbb{Z}_2} \widehat{\mathrm{HFK}}(m(K_2)) \to \widehat{\mathrm{HFK}}(m(K_1 \# K_2))$$

which maps $\widehat{\lambda}_+(L_1) \otimes \widehat{\lambda}_+(L_2)$ to $\widehat{\lambda}_+(L_1 \# L_2)$. Similar statement holds for the $\widehat{\lambda}_-$-invariant. □

Similar results hold for the θ-invariant of transverse knots:

Corollary 2.3 (Vértesi [29]). *Let T_1 and T_2 be transverse knots of topological type K_1 and K_2. Then there are isomorphisms*

$$\mathrm{HFK}^-(m(K_1)) \otimes_{\mathbb{Z}_2[U]} \mathrm{HFK}^-(m(K_2)) \to \mathrm{HFK}^-(m(K_1 \# K_2))$$

and

$$\widehat{\mathrm{HFK}}(m(K_1)) \otimes_{\mathbb{Z}_2} \widehat{\mathrm{HFK}}(m(K_2)) \to \widehat{\mathrm{HFK}}(m(K_1 \# K_2))$$

which map $\theta(T_1) \otimes \theta(T_2)$ *to* $\theta(T_1 \# T_2)$ *and* $\widehat{\theta}(T_1) \otimes \widehat{\theta}(T_2)$ *to* $\widehat{\theta}(T_1 \# T_2)$*, respectively.* □

As an application of the above result we prove:

Theorem 2.4 (Vértesi [29])**.** *There exist infinitely many transversely non-simple knots.*

The definition of the contact invariant in Heegaard Floer homology admits a generalization for Legendrian and transverse knots $\widehat{\mathcal{L}}$ in the knot Floer homology (Lisca-Ozsváth-Stipsicz-Szabó [13]). The contact invariant of Honda, Kazez and Matic for the complement of a Legendrian knot gives rise to a Legendrian invariant: the EH-class. With Stipsicz we understood the relation between these two invariants:

Theorem 2.5 (Stipsicz-Vértesi [27])**.** *There is a map from the sutured Floer homology for the knot-complement to the knot Floer homology mapping* $\widehat{\mathcal{L}}$ *to* EH*.*

A nice consequence of this theorem, which was independently obtained by Vela-Vick [28], is the following:

Theorem 2.6 (Stipsicz-Vértesi [27])**.** *If the knot complement contains Giroux torsion, then* $\widehat{\mathcal{L}}$ *vanishes.*

3. Future directions

The usual way of classifying Legendrian representatives of a knot type consist of three steps. The first one is to prove, that any knot with non-maximal Thurston-Bennequin number is gotten from one with maximal Thurston-Bennequin number by a sequence of well-understood operations called stabilizations. This is not true for any knot type (Etnyre-Honda [6]), and can be subtle to prove. The second step is to understand the maximal Thurston-Bennequin representatives, and at last one needs to understand the relation between the stabilizations of the maximal Thurston-Bennequin representatives. The only transversally non-simple knot type with a complete classification is the $(2,3)$-cable of the $(2,3)$ torus knot (Etnyre-Honda [6]). Using convex surface theory recently with J. Etnyre we managed to understand Legendrian representations of open braids. These techniques should allow us to give a complete classification of Legendrian representatives of positive braids, braids with few (≤ 3) strands. This idea on its own can only be used for knots satisfying the first condition, and as it cannot distinguish Legendrian knots it can only give an upper bound for knot types that are non Legendrian simple. For a complete classification one needs to use other tools as well. Using contact homology and Heegaard Floer homology with Ng we hope to give a complete classification of twist (aka. Chekanov) knots and maybe of two bridge knots.

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